Solitary waves in the excitable Burridge-Knopoff Model

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I. Introduction

- The Burridge-Knopoff model (BK) has been originally introduced to capture certain nonlinear mechanisms of earthquake fault dynamics through a simple nonlinear lattice model. The BK model describes more generally the interaction of two elastic media under frictional contact, one being pulled at constant speed V. This medium is discretized as a chain of blocks (see Fig. 1)

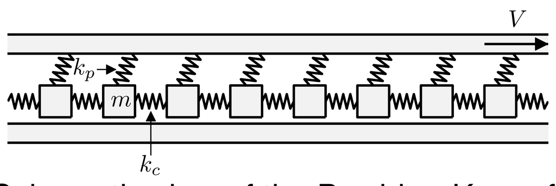


Fig.1 Schematic view of the Burridge-Knopoff model

Each block is subject to a friction force $F(v)$, where v denotes the sliding velocity.

- Friction experiments with metals, rocks, hydrogels, polymers suggest the existence of non-monotonic friction forces increasing at high velocities (see Fig. 2).
- For non-monotonic friction forces, the BK model behaves like an *elastic excitable medium* and supports travelling waves.
- **Overall purpose**: Study the existence of solitary waves (i.e. localized travelling waves) depending on model parameters.

II. The model

- The BK model is expressed by the following equations:

$$\gamma \frac{du_n}{dt} = k\Delta y_n - F(V+u_n) - y_n,$$
$$\frac{dy_n}{dt} = u_n$$

where $\Delta y_n = y_{n+1} - 2y_n + y_{n-1}$ denotes the discrete Laplacian, γ a mass parameter, k the coupling parameter, $F(v)$ the friction function, u_n the velocity variable and y_n the position variable.

- A class of non-monotonic friction forces was explored to probe solitary wave existence. Fig. 3 and Fig. 4 present two of the main friction functions considered, a cubic friction function and an idealized piecewise linear friction function, respectively.

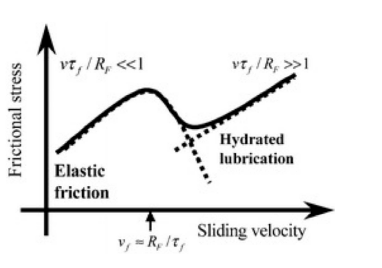


Fig. 2: Friction of a gel adhesive to the substrate in a liquid (Gong, '06').

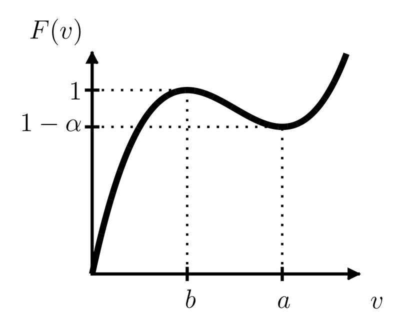


Fig. 3: Smooth friction function.

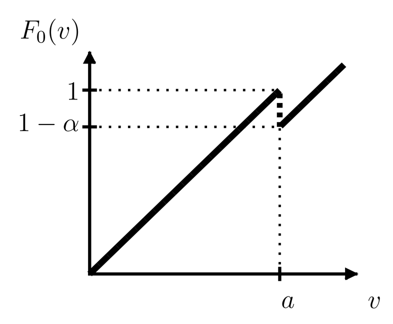


Fig. 4: Non-smooth friction function.

- Our interest is to find solutions $u_n(t) = \varphi(n-ct)$ and $y_n(t) = \psi(n-ct)$ that describe solitary waves. The functions φ, ψ satisfy the advance-delay differential equation:

$$-c\gamma\varphi'(\xi) = k\left[\psi(\xi+1) - 2\psi(\xi) + \psi(\xi-1)\right] - F(V+\varphi(\xi)) - \psi(\xi),$$
$$-c\tilde{\psi}'(\xi) = \varphi(\xi).$$

III. Travelling pulse solutions

The piecewise linear friction function in Fig. 4 is defined by

$$F(v) = \frac{v}{a} - \alpha H(v-a)$$

where $H(x)$ is the Heaviside function. For this friction force, we look for a solution $\varphi(\xi)$ that satisfies:

$$(H): \begin{cases} \varphi(0) = \varphi(\xi_1) = -\beta \\ \varphi(\xi) > -\beta \ \text{ if } \ \xi \in]-\infty, \xi_1[\cup]0, +\infty[\\ \varphi(\xi) < -\beta \ \text{ if } \ \xi \in]\xi_1, 0[, \end{cases}$$

where $\beta = V - a$ and $|\xi_1| =$ solitary wave width.
The resulting linear advance-delay differential equation

$$c^2\gamma\varphi'' - \frac{c}{a}\varphi' + \varphi = k\Delta\varphi + ac(\delta_{\xi_1} - \delta_0)$$

can be solved by Fourier transform:

$$\varphi(\xi; c, \xi_1, k) = \int_{\mathbb{R}} e^{i2\pi\lambda\xi} g_{c,\xi_1,k}(\lambda)d\lambda,$$

where:

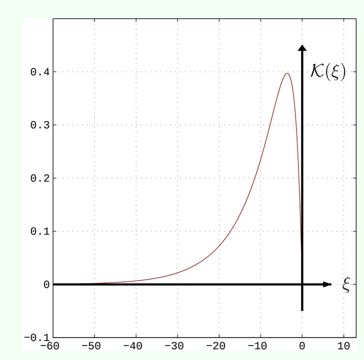
$$g_{c,\xi_1,k}(\lambda) = \alpha c(1 - e^{-2\pi i\lambda\xi_1})\left[4\pi^2c^2x^2\gamma - 4k\sin^2(\pi x) + \frac{2\pi icx}{a} - 1\right]^{-1}$$

We solve system (H) in the overdamped regime, for a large enough wave velocity c, small coupling constant k and $V \approx a$. This leads to the following:

Existence theorem for solitary waves

Assume $\gamma < \frac{1}{4a^2}$. Fix $c > \frac{b}{a\gamma\ln\frac{1+b}{1-b}}$ with $b = \sqrt{1-4a^2\gamma}$. Then for all k small enough, for $V = \bar{V}(k) = a + \mathcal{O}(k) > a$, $\exists$ a *solitary wave solution* of the BK model taking the following form when $k \to 0$:

$$\varphi(\xi) = \alpha c[\mathcal{K}(\xi - \xi_1) - \mathcal{K}(\xi)] + \mathcal{O}(k) \quad \text{Solution of BK model}$$
$$\xi_1 = \mathcal{O}(\ln k) \quad \text{Solitary wave width}$$



$$\left(c^2\gamma\frac{d^2}{d\xi^2} - \frac{c}{a}\frac{d}{d\xi} + 1\right)\mathcal{K} = \delta_0$$

Fig. 5: Shape of function $\mathcal{K}(\xi)$

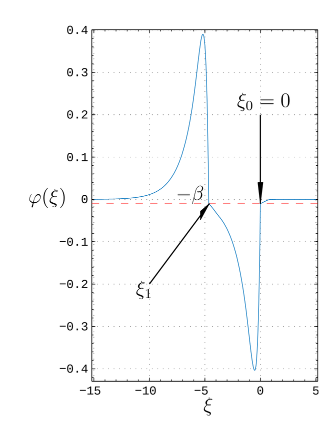


Fig. 6

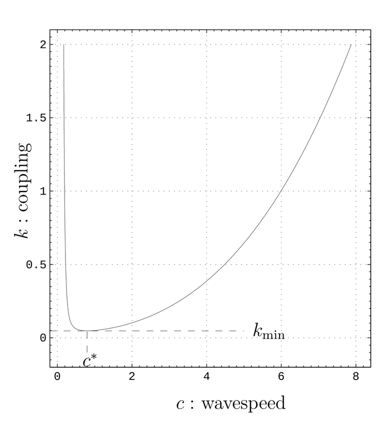


Fig. 7

- A solitary wave solution is shown in Fig. 6 supported by the existence theorem.
- Solitary waves satisfying (H) exist only above a minimal coupling strength $k_{\min} = \mathcal{O}(V-a)$. Fig. 7 presents the coupling strength k as a function of the wave speed c of the solitary wave.

IV. Dynamical simulations

- The propagation of a stable solitary wave is shown in Fig. 8. There exist solitary waves with oscillatory tails for $V \to a$ as shown in Fig. 9. Certain non-monotonic friction forces also lead to bistability between the ground state and periodic oscillations, see Fig. 10. If the continuum limit $k \to +\infty$ is approached, solitary waves develop oscillatory shocks during a long transitory regime, see Fig. 11.

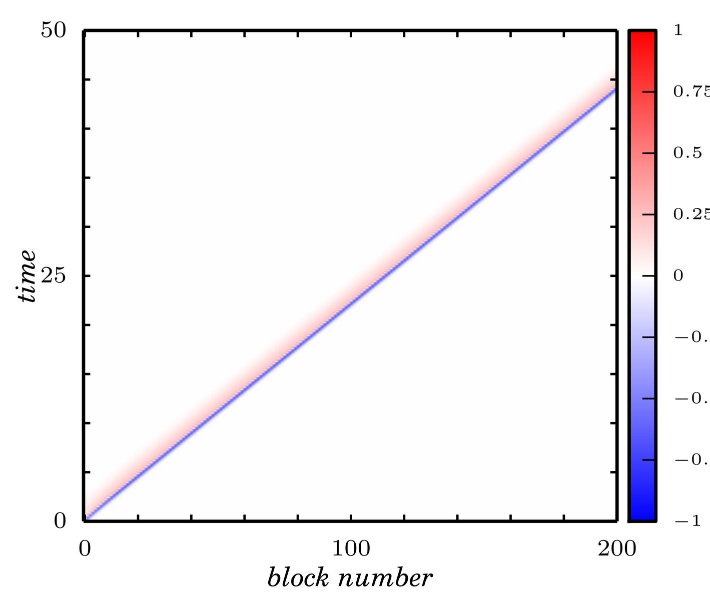


Fig. 8: Velocities $u_n(t)$.

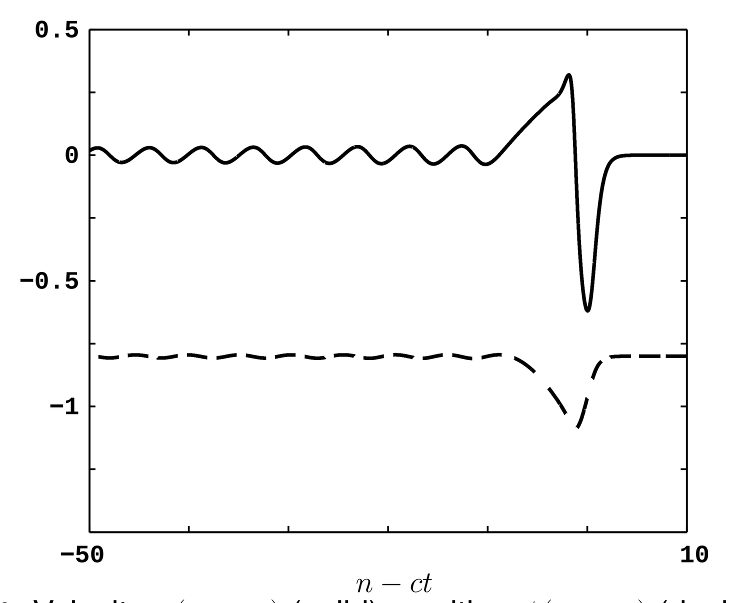


Fig. 9: Velocity $\varphi(n-ct)$ (solid), position $\psi(n-ct)$ (dashed).

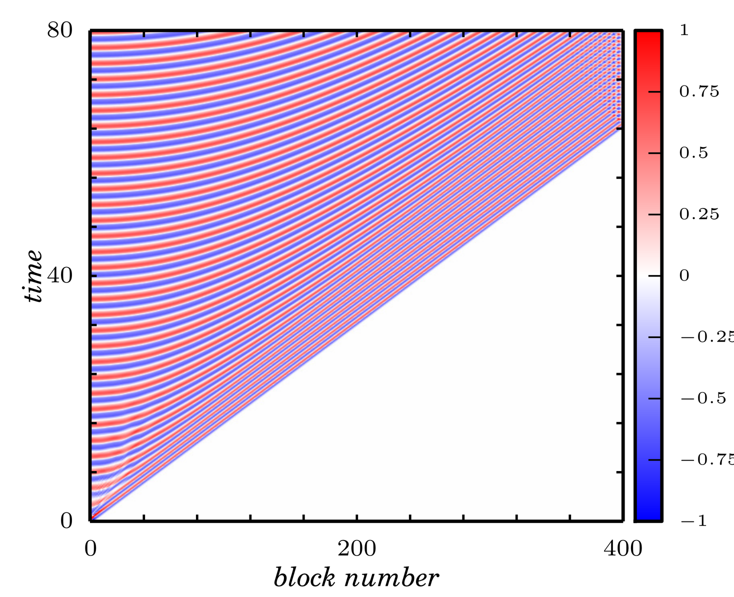


Fig. 10: Velocities $u_n(t)$.

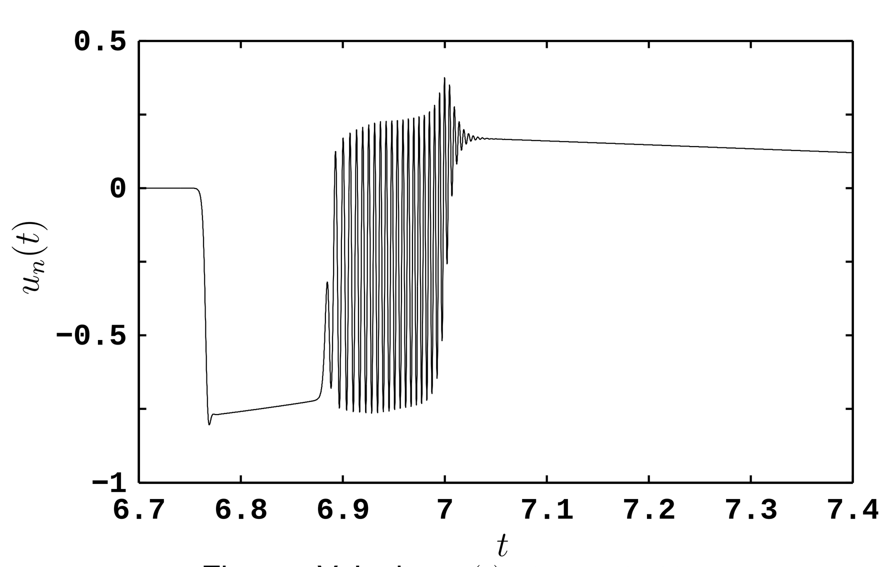


Fig. 11: Velocity $u_n(t)$ at $n = 17500$

V. Conclusions

- In the excitable Burridge Knopoff model, we report for the first time the existence of solitary waves for a wide range of non-monotonic cubic-like friction forces and parameters. For a piecewise-linear and discontinuous friction force, we proved existence of solitary waves at small coupling k for $V \approx a$.
- Besides the classical earthquake fault interpretation, the excitable BK equations may find interesting applications for the description of friction at the interface of patterned surfaces, where the blocks correspond to discrete surface elements. Other applications involve mechanical oscillator networks and nonlinear electric transmission lines.